

GEODESICS AND SUBMANIFOLD STRUCTURES IN CONFORMAL GEOMETRY

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ABSTRACT. A conformal structure on a manifold M^n induces natural second order conformally invariant operators, called Möbius and Laplace structures, acting on specific weight bundles of M , provided that $n \geq 3$. By extending the notions of Möbius and Laplace structures to the case of surfaces and curves, we develop here the theory of extrinsic conformal geometry for submanifolds, find tensorial invariants of a conformal embedding, and use these invariants to characterize various forms of geodesic submanifolds.

2010 *Mathematics Subject Classification:* Primary 53A30, 53A55.

Keywords: Conformal structure, Laplace structure, Möbius structure, Schouten-Weyl tensor, fundamental form, conformal invariant, geodesic.

1. INTRODUCTION

The existence of a unique covariant derivative makes differential calculus, including the concept of totally geodesic submanifolds (or, more generally, the tensorial invariants of a Riemannian embedding) in a Riemannian manifold straightforward.

On a conformal manifold, where there is no such canonical covariant derivative (there is, however, a *Cartan connection* on an enlarged bundle, for dimension at least 3, [8]), a concept of *conformal geodesics* is also given [10], [12] starting from dimension 3 onwards.

These *conformal geodesics* are curves that are solutions of a 3rd order ODE that depends on the conformal structure alone.

In this paper, we intend to characterize higher-dimensional submanifolds that fulfil some *geodesic* properties in the conformal setting and describe the geometric properties and invariants of a conformal embedding.

To make the theory fully general, a first inconsistency of conformal geometry has to be overcome: indeed, while in dimensions larger than 3 a conformal manifold admits an associated Cartan connection and is, therefore, *rigid*, on curves, a conformal structure means just a differential structure, and on surfaces, a conformal structure and an orientation are equivalent to a complex structure – both are examples of *flexible structures*.

Here, we call a geometric structure on M *rigid* if on every open set, the dimension of the space of infinitesimal transformations (vector fields) preserving the given structure is bounded by a number that depends only on $\dim M$ and the corresponding structure. Otherwise it is *flexible*. Examples of rigid structures are Riemannian metrics, conformal structures if $\dim M \geq 3$, *CR* structures and, more generally, all structures that admit a canonical Cartan connection; symplectic, complex, contact structures are examples of flexible structures.

Using the concept of a *Möbius structure*, defined by D. Calderbank as a linear second order differential operator of a certain type [7], and also using a *Laplace structure* (a variant of the *conformal Laplacian*) to rigidify a curve, we create a setting conformal-Möbius-Laplace for which the questions of submanifold geometry can be studied without conditions on the dimension.

In particular, on a Möbius surface or a Laplace curve, the concept of a conformal (or rather Möbius, resp. Laplace) geodesic is well-defined, and denotes, as well, the family of curves that are solutions to a 3rd order ODE. Technically, these equations are given, in terms of a conformal covariant derivative (*Weyl structure*) and of its associated *Schouten-Weyl tensor*, and this tensor is defined, in low dimensions, precisely by the corresponding additional structure (Möbius, resp. Laplace). The invariants of, and induced structures on a conformal embedding are also defined in terms of these Schouten-Weyl tensors, the distinction between them being the following:

- An intrinsic structure is one that can be defined and considered in terms of the submanifold alone, without any reference to the embedding: it is the case of the induced conformal, Möbius and Laplace structures.
- An extrinsic kind of structure refers explicitly to (some infinitesimal version - like the normal bundle - of) the embedding of the submanifold in its conformal (or Möbius) ambient space: it is the case of some tensorial invariants of the embedding and of the induced connection on the weightless normal bundle.

Geometrically, a Laplace structure on a curve is a *projective structure* [15], [14] and the global projective geometry of a closed curve in a conformal (or Möbius) ambient space turns out to be a very interesting, and largely unknown problem, as a forthcoming paper shows [4].

After a preliminary section where we recall some basic facts of conformal geometry (weight bundles, Weyl structures, and curvature decompositions), with a particular focus on low dimensions, we recall in Section 3 the definition of the conformal geodesic equation and define certain tensorial invariants of a conformal embedding.

In the 4th Section, the Möbius and Laplace structures turn out to be implicit on a conformal manifold of dimension at least 3, and they only need to be specified explicitly in low dimensions. On the other hand, embedded submanifolds in higher-dimensional conformal manifolds turn out to admit induced Möbius, resp. Laplace structures. Relating these induced structures to the implicit ones (if the submanifold has dimension at least 3) leads again to some of the tensorial invariants of Section 3.

Theorem 4.22 shows that, besides the particular case of hypersurfaces, when one of the invariants vanishes identically, as a consequence of the Gauß-Codazzi equations, any given such tensorial objects on a given manifold and on its normal bundle can be realized as the invariants of an embedding in some ambient space.

In Section 5, we show that these *invariant tensors* of an embedding turn out to be obstructions for various properties that generalize, in the conformal context, the *totally geodesic submanifolds* of Riemannian geometry.

More precisely, a submanifold is called *totally umbilic* if it is totally geodesic for some metric in the conformal class, it is *weakly geodesic* if it is spanned by conformal geodesics in the ambient space, and *strongly geodesic* if its conformal geodesics are also conformal geodesics

in the ambient space (for dimensions 1 or 2, the conformal structure of the (sub)manifold needs to be completed (for rigidity) by a Laplace, resp. Möbius structure).

Finally, these different kinds of geodesic properties of a submanifold are shown to satisfy some implications (among which the fact that *strongly geodesic* implies *weakly geodesic* turns out to be non-trivial), and can be characterized by the vanishing of some of the above mentioned tensorial invariants, Theorem 5.4.

2. PRELIMINARIES ON CONFORMAL GEOMETRY

In this section, we review the main notions needed in conformal geometry. Good references are [2] and [13], however we need to push some of the formulas beyond their usual lower bound for the dimension, like in [7] (in particular for the Schouten-Weyl tensor and the normalized scalar curvature); a reader familiar with the formalism of weight bundles, Weyl structures, etc., may jump directly to Proposition 2.11.

2.1. Weight bundles. Let M be a m -dimensional manifold with density bundle $|\Lambda|M$. This is an oriented line bundle, hence trivial, whose positive sections are the *volume elements* of M , allowing the integration of functions on the manifold; it is isomorphic, if M is oriented, with $\Lambda^m M$, the bundle of m -forms on M (the isomorphism depends on the orientation).

A *conformal structure* on M is a *positive-definite* symmetric bilinear form c on TM with values in the line bundle $L^2 := L \otimes L$, or, equivalently, a non-degenerate section $c \in C^\infty(S^2 M \otimes L^2)$ (Here we denote by $S^2 M$ the bundle of symmetric bilinear forms on TM), with the following normalization condition:

$$|\det c| : (\Lambda^m TM)^2 \rightarrow (L^2)^m$$

is the identity. (Note that $(\Lambda^m M)^2 \simeq (|\Lambda|M)^2 \simeq L^{2m}$.)

Remark 2.1. Each positive section l of L trivializes it, hence

$$g_l := l^{-2} c : TM \otimes TM \rightarrow \mathbb{R}$$

is a Riemannian metric on M . If $l' := e^f l$ is another positive section (for $f : M \rightarrow \mathbb{R}$ a smooth function), then the metric $g_{l'} = e^{-2f} g_l$ is conformally equivalent to g_l , and they belong to the same conformal class, defined by c .

Remark 2.2. Because L is a trivial bundle, not only natural powers (defined as multiple tensor products: $L^k = L \otimes \dots \otimes L$) or negative integer powers (L^{-1} is the dual of L and $L^{-k} := L^* \otimes \dots \otimes L^*$) are well-defined, but also *real* powers of L : indeed, the bundle L^k , $k \in \mathbb{R}$, is the bundle associated to the frame bundle $GL(M)$ and the representation $|\det|^{\frac{k}{n}}$.

Remark 2.3. A conformal structure is equivalent to a reduction of $GL(m)$ to the $CO(m) = O(m) \times \mathbb{R}_+^*$ -bundle of conformal frames $CO(M)$.

Convention. The usual identifications of vectors and covectors from Riemannian geometry can be applied in the conformal setting, but it involves a tensor product with a corresponding weight bundle: $T^*M \simeq TM \otimes L^{-2}$. In general, an irreducible representation of $CO(m)$ is the tensor product of an irreducible representation of $O(m)$ and one of \mathbb{R}_+^* , the latter being the multiplication by the k th power of an element of \mathbb{R}_+^* , where $k \in \mathbb{R}$ is called the *conformal weight* of the representation. In particular, for the associated bundles, TM has conformal weight 1,

T^*M has conformal weight -1 , a k -weighted (r, s) tensor bundle $A \subset \otimes^r T^*M \otimes \otimes^s TM \otimes L^k$ has weight $s - r + k$ (in particular, any endomorphism bundle has conformal weight zero). Two irreducible bundles are isomorphic as $\text{CO}(M)$ -bundles if and only if they are isomorphic as $\text{O}(M)$ -bundles (for any metric in the conformal class) *and* they have the same conformal weight.

2.2. Weyl structures. Unlike in (semi-) Riemannian geometry, a conformal manifold does not carry a canonical affine connection. Instead, there is a family of adapted connections, the *Weyl structures*:

Definition 2.4. A Weyl structure ∇ on a conformal manifold (M, c) is a torsion-free, *conformal* connection on TM , i.e. $\nabla c = 0$.

The fundamental theorem of conformal geometry can now be stated:

Theorem 2.5. [20] *Let (M, c) be a conformal manifold and denote, for a Weyl structure ∇ , by ∇^L the connection induced by ∇ on L . The correspondence*

$$\{\text{Weyl structures on } M\} \longrightarrow \{\text{connections on } L\},$$

given by $\nabla \mapsto \nabla^L$ is one-to-one.

More precisely, the inverse map is given by the following *conformal Koszul formula* [7], [13], [20]:

$$2c(\nabla_X Y, Z) = \nabla_X^L(c(Y, Z)) + \nabla_Y^L(c(X, Z)) - \nabla_Z^L(c(X, Y)) + c([X, Y], Z) + c([X, Z], Y) - c([Y, Z], X). \quad (1)$$

A consequence of Theorem 2.5 and (1) is the relation between two Weyl structures: as the difference between two linear connections ∇'^L and ∇^L on the line bundle L is a 1-form θ , the difference between the corresponding Weyl structures ∇' and ∇ must be given by a tensor $\tilde{\theta}$ that depends linearly on θ . More precisely, applying (1) we get:

$$\nabla'_X Y - \nabla_X Y = \tilde{\theta}_X Y := (\theta \wedge X)(Y) + \theta(X)Y, \quad (2)$$

where $\theta \wedge X$, the wedge product of a 1-form and a vector, is the skew-symmetric endomorphism of TM defined by

$$(\theta \wedge X)(Y) := \theta(Y)X - c(X, Y)\theta. \quad (3)$$

Here, note that the 1-form θ is a section of $T^*M \simeq TM \otimes L^{-2}$ and thus $c(X, Y)\theta$ is a section of $TM \otimes L^{-2} \otimes L^2 \simeq TM$.

Remark 2.6. The difference tensor $\tilde{\theta}_X$ from (2) lies in the adjoint bundle $\mathfrak{co}(M)$ of the bundle of conformal frames $\text{CO}(M)$, and thus the difference of the induced connections by ∇, ∇' satisfying (2) on some (weighted) tensor bundle E of conformal weight $k \in \mathbb{R}$ is given by

$$\nabla'_X \xi - \nabla_X \xi = \tilde{\theta}_X \xi = (\theta \wedge X)\xi + k\theta(X)\xi,$$

where $(\theta \wedge X)\xi$ is the usual action of skew-symmetric endomorphisms on a tensor ξ , and $k\theta(X)\xi$ is the Lie algebra action of $\theta(X)\text{Id}$ on a representation of conformal weight k .

For example, for a section l of L^k we have

$$\nabla'_X l - \nabla_X l = k\theta(X)l, \quad (4)$$

and, for an endomorphism $A : TM \rightarrow TM$, we have

$$\nabla'_X A - \nabla_X A = [\theta \wedge X, A], \quad (5)$$

where the square bracket is the commutator of endomorphisms..

2.3. Curvature. The curvature of a Weyl structure ∇ is defined by

$$R_{X,Y}^\nabla Z := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z,$$

and can be seen as a 2-form with values in $\mathfrak{co}(M)$. The identity component of this 2-form is the *Faraday* form $F^\nabla \in \Lambda^2 M$, which is the curvature of the connection ∇^L , more precisely

$$R_{X,Y}^\nabla = (R_{X,Y}^\nabla)^{skew} + F^\nabla \otimes \text{Id}.$$

R^∇ satisfies the Bianchi identities (tensorial and differential), the tensorial (or the *first*) Bianchi identity being:

$$R_{X,Y}^\nabla Z + R_{Y,Z}^\nabla X + R_{Z,X}^\nabla Y = 0, \quad \forall X, Y, Z \in TM. \quad (6)$$

Definition 2.7. The *suspension* of a bilinear form $A \in T^*M \otimes T^*M$ by the identity is a 2-form $A \wedge \text{Id}$ with values in $\mathfrak{so}(M)$, given by

$$(A \wedge \text{Id})_{X,Y} := A(Y, \cdot) \wedge X - A(X, \cdot) \wedge Y, \quad (7)$$

where we note, as in (3), that the wedge product between a vector and a 1-form (here $A(Y, \cdot)$) is a skew-symmetric endomorphism.

The tensorial Bianchi identity (6) is satisfied by the tensor

$$\tilde{F}^\nabla := -\frac{1}{2}F^\nabla \wedge \text{Id} + F^\nabla \otimes \text{Id}. \quad (8)$$

If we define the *Riemannian component* of R^∇ as

$$R_{X,Y}^{\nabla, Riem} := (R_{X,Y}^\nabla)^{skew} + \frac{1}{2}F^\nabla \wedge \text{Id},$$

then it is a 2-form with values in $\mathfrak{so}(M)$ that satisfies the first Bianchi identity (like a Riemannian curvature tensor).

Remark 2.8. The linear map

$$\text{End}(TM) \ni A \longmapsto A \wedge \text{Id} \in \Lambda^2 M \otimes \mathfrak{so}(M)$$

is injective for $n \geq 3$, and its kernel consists of all trace-free endomorphisms if $n = 2$. On the other hand, every tensor in $\Lambda^2 M \otimes \mathfrak{co}(M)$ satisfies the Bianchi identity for $n = 2$, hence the correction term $-\frac{1}{2}F^\nabla \wedge \text{Id}$ (which is actually zero) is not needed there.

In general, the *Ricci contraction* of a curvature tensor is the map

$$\text{Ric} : \Lambda^2 M \otimes \mathfrak{co}(M) \rightarrow \text{End}(TM),$$

$$\text{Ric}(R)(X, Y) := \text{tr}(R_{\cdot, X} Y),$$

and the *Ricci tensor* of ∇ is $\text{Ric}^\nabla := \text{Ric}(R^\nabla)$.

A straightforward computation shows

$$\text{Ric}(A \wedge \text{Id}) = (n-2)A + \text{tr}_c A \cdot c. \quad (9)$$

This implies

$$\text{Ric}(\tilde{F}^\nabla) = -\frac{n-2}{2}F^\nabla - F^\nabla = -\frac{n}{2}F^\nabla.$$

On the other hand, the Ricci contraction, applied to the Riemannian component $R^{\nabla, \text{Riem}}$ of R^∇ , produces a symmetric tensor

$$\text{Ric}^{\nabla, s} := \text{Ric}(R^{\nabla, \text{Riem}}) \in S^2 M.$$

Therefore, the Ricci tensor $\text{Ric}^\nabla := \text{Ric}(R^\nabla)$ of a Weyl structure ∇ has a skew-symmetric part, equal to $-n/2 F^\nabla$, and a symmetric part, equal to $\text{Ric}^{\nabla, s} = \text{Ric}(R^{\nabla, \text{Riem}})$.

The relation (9) also implies that, if $n \geq 3$, then, for every given bilinear form $A \in \otimes^2 M$, there exists another bilinear form $h(A) \in \otimes^2 M$ such that $\text{Ric}(h(A) \wedge \text{Id}) = A$.

Straightforward computations show that the linear map $h : \otimes^2 M \rightarrow \otimes^2 M$ has the expression

$$h(A) = \frac{1}{n-2}A_0^s + \frac{1}{2n(n-1)}\text{tr}_c A \cdot c - \frac{1}{2}A^{\text{skew}}, \quad (10)$$

where $A_0^s \in S_0^2 M$ is the symmetric, trace-free part of A , $\text{tr}_c A \in L^{-2}$ is the trace of A w.r.t. c , and $A^{\text{skew}} \in \Lambda^2(M)$ is the skew-symmetric part of A .

Definition 2.9. The *scalar curvature* of an Weyl structure ∇ on the conformal manifold (M, c) is the density of weight -2

$$\text{Scal}^\nabla := \text{tr}_c(\text{Ric}^\nabla), \quad (11)$$

given by the trace (with respect to c) of the Ricci tensor of ∇ .

If $n := \dim M \geq 3$, the *Schouten-Weyl tensor* h^∇ of a Weyl structure ∇ on (M, c) is the tensor

$$h^\nabla := h(\text{Ric}^\nabla) \in \otimes^2 M. \quad (12)$$

Its symmetric part $h^{\nabla, s} := h(\text{Ric}^{\nabla, s})$ is called the symmetric Schouten-Weyl tensor of ∇ .

In particular, for ∇ the Levi-Civita connection of the metric g , we have the well-known formula

$$h^g = \frac{1}{n-2}\text{Ric}_0^g + \frac{1}{2n(n-1)}\text{Scal}^g \cdot g. \quad (13)$$

Remark 2.10. From Remark 2.8 it follows that, even if the Schouten-Weyl tensor on ∇ is only defined if $n \geq 3$, there always exist bilinear forms $h \in \otimes^2 M$ such that $\text{Ric}(h \wedge \text{Id}) = \text{Ric}^\nabla$: for $n \geq 3$ this requires h to be the Schouten-Weyl tensor h^∇ as defined in (12); for $n \geq 2$ the skew-symmetric part of h has to be $-\frac{1}{2}F^\nabla$ and the pure trace part has to be $\frac{1}{4}\text{Scal}^\nabla \cdot c \in S^2 M$ (the trace-free part of h is undetermined if $n = 2$); and for $n = 1$ h is an undetermined function. We can therefore define

$$\sigma^\nabla := \frac{1}{2(n-1)}\text{Scal}^\nabla \in L^{-2} \quad (14)$$

as the *normalized scalar curvature*, for $n \geq 2$, and $\frac{1}{n}\sigma^\nabla \cdot c$ is the pure trace part of any of the tensors h above (in particular, if $n \geq 3$, σ^∇ is the trace of the Schouten-Weyl tensor h^∇).

The Schouten-Weyl tensor is sometimes called the normalized Ricci tensor; it is equal to k times the metric on every Riemannian manifold of constant sectional curvature k .

After subtracting from $R^{\nabla, Riem}$ the suspension of $h^{\nabla, s}$, we obtain a tensor $W^{\nabla} \in \Lambda^2 M \otimes \mathfrak{so}(M)$ that satisfies the Bianchi identity, and is trace-free (up to a constant, the only non-trivial contraction (or trace) on the space of the Riemannian curvature tensors is the Ricci contraction). This tensor is called the *Weyl tensor* of ∇ .

The following decomposition is a direct consequence of (9) and of the definition above:

Proposition 2.11. *The curvature R^{∇} of a Weyl structure ∇ on a conformal manifold (M, c) of dimension $n \geq 3$ decomposes as*

$$R^{\nabla} = h^{\nabla} \wedge \text{Id} + W + F^{\nabla} \otimes \text{Id}, \quad (15)$$

where W^{∇} is the Weyl tensor, F^{∇} is the Faraday 2-form, and

$$h^{\nabla} = h^{\nabla, s} - \frac{1}{2} F^{\nabla}$$

is the full Schouten-Weyl tensor of ∇ .

Note that $W = 0$ if $n = 3$ by dimension reasons.

If $n = 2$, the curvature decomposition is even simpler, since $\Lambda^2 M \otimes \mathfrak{co}(M)$ is 2-dimensional (and automatically satisfies the Bianchi identity):

$$R^{\nabla} = \frac{1}{2} \sigma^{\nabla} \cdot c \wedge \text{Id} + F^{\nabla} \otimes \text{Id},$$

where $\sigma^{\nabla} = \frac{1}{2} \text{Scal}^{\nabla}$ is a section of L^{-2} .

We conclude that the curvature tensor R^{∇} of a conformal manifold (M, c) of dimension n is determined by

- (1) its Faraday curvature $F^{\nabla} \in \Lambda^2 M$ (for $n \geq 2$)
- (2) its scalar curvature $\text{Scal}^{\nabla} := \text{tr}_c \text{Ric}^{\nabla} \in L^{-2}$, or, equivalently, its normalized scalar curvature $\sigma^{\nabla} = \frac{1}{2(n-1)} \text{Scal}^{\nabla}$ (for $n \geq 2$)
- (3) its trace-free symmetric Ricci tensor $\text{Ric}_0^{\nabla, s} \in S_0^2 M$ (for $n \geq 3$)
- (4) its trace-free part W^{∇} , the Weyl tensor (for $n \geq 4$).

All these components are sections in $\text{CO}(n)$ -irreducible vector bundles.

We give now the transformation rule for the curvature tensors corresponding to two Weyl structures, more precisely, the transformation rules for their $\text{CO}(n)$ irreducible components in the list above:

Proposition 2.12. *For two Weyl structures $\nabla' = \nabla + \tilde{\theta}$ on a conformal manifold (M, c) of dimension $n \geq 3$, the corresponding Schouten-Weyl tensors are related by:*

$$h^{\nabla'} - h^{\nabla} = -\nabla \theta + \theta \otimes \theta - \frac{1}{2} c(\theta, \theta) c. \quad (16)$$

Moreover, the Weyl tensor W is independent on the Weyl structure and depends on the conformal structure only. The Faraday curvature changes as follows:

$$F^{\nabla'} = F^{\nabla} + d\theta.$$

Proof. Let us compute the curvature of ∇' by deriving (2), and using (5) at a point where the ∇ -derivatives of the involved vector fields vanish:

$$\nabla'_X \nabla'_Y Z = \nabla_X \nabla_Y Z + (\nabla_X \theta \wedge Y)(Z) + (\tilde{\theta}_X) \circ (\tilde{\theta}_Y)(Z) + (\nabla_X \theta)(Y)Z, \quad (17)$$

Therefore

$$R_{X,Y}^{\nabla'} Z - R_{X,Y}^{\nabla} Z = (\nabla_X \theta \wedge Y)(Z) - (\nabla_Y \theta \wedge X)(Z) + d\theta(X, Y)Z + [\tilde{\theta}_X, \tilde{\theta}_Y](Z).$$

Note that $[\tilde{\theta}_X, \tilde{\theta}_Y] = [\theta \wedge X, \theta \wedge Y]$, hence we get

$$R_{X,Y}^{\nabla'} Z = R_{X,Y}^{\nabla} Z - (\nabla \theta \wedge \text{Id})_{X,Y} Z + [\theta \wedge X, \theta \wedge Y](Z) + d\theta(X, Y)Z.$$

We compute directly

$$\begin{aligned} [\theta \wedge X, \theta \wedge Y] &= (\theta \otimes \theta)(Y) \wedge X - (\theta \otimes \theta)(X) \wedge Y + c(\theta, \theta)X \wedge Y \\ &= \left(((\theta \otimes \theta) \wedge \text{Id})_{X,Y} - \frac{1}{2}c(\theta, \theta)(c \wedge \text{Id})_{X,Y} \right), \end{aligned}$$

that implies

$$F^{\nabla'} = F^{\nabla} + d\theta, \quad W^{\nabla'} = W^{\nabla}$$

and the result claimed in (16). \square

Corollary 2.13. *Let ∇, ∇' be Weyl structures on the conformal manifold (M, c) of dimension at least 3, such that (2) holds. The relations between the trace-free parts, resp. the traces of the Schouten-Weyl tensors of ∇ and ∇' are:*

$$h_0^{\nabla'} = h_0^{\nabla} - (\nabla \theta)_0 + (\theta \otimes \theta)_0, \quad (18)$$

$$\sigma^{\nabla'} = \sigma^{\nabla} + \delta^{\nabla} \theta + \frac{2-n}{2}c(\theta, \theta). \quad (19)$$

Here $\delta^{\nabla} \theta := -\text{tr}_c \nabla \theta = -\sum_{i=1}^n (\nabla_{e_i} \theta)(e_i)$, for $\{e_i\}$ an c -orthonormal basis of TM .

Remark 2.14. Proposition 2.12 holds regardless of the dimension n of M in the following sense: *assuming* that the curvature tensors R^{∇} has the expression (15), for *some* tensor h^{∇} (of Schouten-Weyl type), then $R^{\nabla'}$ has also an expression (15), with $h^{\nabla'}$ given by (16). As mentioned in Remark 2.10, if $n = 2$ only the skew-symmetric part, and the pure trace part of h^{∇} are determined by the Ricci tensor Ric^{∇} . The transformation rule (19) is, thus, also valid for $n = 2$.

Note, however, that even if the curvature of a Weyl structure does neither define a tensor of type h_0^{∇} in $S_0^2 M$ for $n = 2$ nor a density of type σ^{∇} in L^{-2} for $n = 1$, these bundles are not zero themselves. We will see that some extra geometric structures on (M, c) induce, for all Weyl structures ∇ , sections h_0^{∇} , resp. σ^{∇} in these bundles, such that the transformation rules from Corollary 2.13 hold.

Indeed, following D. Calderbank [7], we introduce the *Möbius differential operator* on a conformal manifold:

3. MÖBIUS AND LAPLACE STRUCTURES ON CONFORMAL MANIFOLDS

Proposition 3.1. [2], [6], [7], [12]. *Let (M, c) be a conformal manifold of dimension $m \geq 3$, and let Hess_0^∇ , h_0^∇ be trace-free Hessian, resp. trace-free Schouten tensor of a Weyl structure ∇ . Then the second order differential operator $\mathcal{M}^\nabla : C^\infty(L) \rightarrow C^\infty(S_0^2 M \otimes L)$ defined by*

$$\mathcal{M}_{(X,Y)}^\nabla l := \text{Hess}_0^\nabla(X, Y)l + h_0^{\nabla, s}(X, Y)l, \quad (20)$$

is independent of ∇ . Here $h_0^{\nabla, s}$ is the symmetric trace-free Schouten-Weyl tensor of ∇ .

Proof. Using (4) we easily get

$$\begin{aligned} \text{Hess}^{\nabla'}(X, Y)l - \text{Hess}^\nabla(X, Y)l &= (k-1)[\theta(X)\nabla_Y l + \theta(Y)\nabla_X l] + c(X, Y)\nabla\theta l \\ &\quad + k[(\nabla_X\theta)(Y) + (k-2)\theta(X)\theta(Y) + c(X, Y)c(\theta, \theta)]l. \end{aligned} \quad (21)$$

Taking $k = 1$ and using (16) shows that $\text{Hess}_0^\nabla(X, Y)l + h_0^{\nabla, s}(X, Y)l$ does not depend on the choice of the Weyl structure ∇ . \square

As an immediate corollary, we see that the difference of the trace-free Hessians associated to two arbitrary Weyl structures is a scalar operator on L . This motivates the following

Definition 3.2. (Calderbank [7]) A *Möbius structure* on a conformal manifold (M, c) of dimension at least 2 is a second order linear differential operator

$$\mathcal{M} : C^\infty(L) \rightarrow C^\infty(S_0^2 M \otimes L)$$

such that $\mathcal{M} - \text{Hess}_0^\nabla$ is a scalar operator for some (and thus all) Weyl structures ∇ on (M, c) .

The Möbius structure defined by (20) is called the *canonical Möbius structure* of (M, c) and will be denoted by \mathcal{M}^c .

Remark 3.3. If $\dim M \geq 3$, then every Möbius structure \mathcal{M} is the sum of the canonical Möbius structure \mathcal{M}^c and a trace-free symmetric bilinear form on M . If $\dim M = 2$, we define $h_0^{\mathcal{M}, \nabla} := \mathcal{M} - \text{Hess}_0^\nabla$ as a symmetric bilinear form and we conclude from (21) that $h_0^{\mathcal{M}, \nabla}$ satisfies the transformation rule (18). In fact, we have:

Proposition 3.4. [7] *A Möbius structure \mathcal{M} on a conformal manifold (M, c) is equivalent to a map*

$$h_0^\mathcal{M} : \{ \text{Weyl structures on } (M, c) \} \rightarrow C^\infty(S_0^2 M)$$

such that if ∇, ∇' are Weyl structures satisfying (2), then $h_0^{\mathcal{M}, \nabla} := h_0^\mathcal{M}(\nabla)$ and $h_0^{\mathcal{M}, \nabla'} := h_0^\mathcal{M}(\nabla')$ satisfy (18).

Remark 3.5. On a Möbius surface, the trace-part of the Schouten-Weyl tensor, σ^∇ , is well-defined for every Weyl structure ∇ (it uses just the underlying conformal structure, see Remark 2.14), and the same holds for the Faraday form (hence for the skew-symmetric part of what should be the Schouten-Weyl tensor). The Möbius structure, in turn, associates to ∇ the symmetric, trace-free *Möbius Schouten-Weyl tensor* $h_0^{\mathcal{M}, \nabla}$ as above. This means that, on a Möbius surface, each Weyl structure has its own Schouten-Weyl tensor, just like in the case of a conformal manifold of higher dimension.

A similar construction (which yields the well-known conformal Laplacian) holds for the trace of the Hessian of Weyl structures. Note that the Schouten-Weyl tensor is only defined for conformal manifolds of dimension $m > 2$, but, in dimension 2, its trace σ^∇ (which is a multiple of the scalar curvature) is still well-defined, and we have the following well-known fact:

Proposition 3.6. *On a conformal manifold of dimension $m \geq 2$, the second order differential operator $\mathcal{L}^\nabla : C^\infty(L^k) \rightarrow C^\infty(L^{k-2})$*

$$\mathcal{L}l = \mathcal{L}^c l := \operatorname{tr}(\operatorname{Hess}^\nabla l) + \left(1 - \frac{m}{2}\right) \sigma^\nabla l,$$

where ∇ is any Weyl structure and $\operatorname{Hess}^\nabla$, σ^∇ are the corresponding Hessian, resp. pure-trace part of Schouten tensor of ∇ , is independent on ∇ for $k = 1 - \frac{m}{2}$.

Proof. This follows directly after taking the conformal trace in (16) and (21). \square

The operator \mathcal{L}^c is the *conformal Laplacian* of the conformal manifold (M, c) , or the *Yamabe operator*.

Proposition 3.6 motivates the following

Definition 3.7. A *Laplace structure* on an m -dimensional conformal manifold (M, c) is a second order linear differential operator

$$\mathcal{L} : C^\infty(L^k) \rightarrow C^\infty(L^{k-2}),$$

where $k := 1 - m/2$, such that $\mathcal{L} - \operatorname{tr}(\operatorname{Hess}^\nabla)$ is a scalar operator for some (and thus all) Weyl structures ∇ on (M, c) .

The Laplace structure defined by Proposition 3.6 is called the *canonical Laplace structure* of (M, c) .

Remark 3.8. If $\dim M \geq 2$, then every Laplace structure \mathcal{L} is the sum of the canonical Laplace structure \mathcal{L}^c and a section of L^{-2} . If $\dim M = 1$ and we have fixed a Laplace structure \mathcal{L} on the curve M , we define $\sigma^{\mathcal{L}, \nabla} := \mathcal{L} - \operatorname{Hess}^\nabla$ as a section of L^{-2} and we conclude from (21) that $\sigma^{\mathcal{L}, \nabla}$ satisfies the transformation rule (19). In fact, as in Proposition 3.4, we immediately get:

Proposition 3.9. [6] *A Laplace structure \mathcal{L} on a conformal manifold (M, c) is equivalent to a map*

$$\sigma^{\mathcal{L}} : \{\text{Weyl structures on } (M, c)\} \rightarrow C^\infty(L^{-2})$$

such that if ∇, ∇' are Weyl structures satisfying (2), then $\sigma^{\mathcal{L}, \nabla} := \sigma^{\mathcal{L}}(\nabla)$ and $\sigma^{\mathcal{L}, \nabla'} := \sigma^{\mathcal{L}}(\nabla')$ satisfy (19).

Remark 3.10. Like in the case of a Möbius surface (Remark 3.5), on a Laplace curve (C, \mathcal{L}) , $\sigma^{\mathcal{L}, \nabla}$ plays therefore the role of the Schouten-Weyl tensor associated to the connection ∇ . In [6], the authors use also the name *Möbius structures* for 1-dimensional Laplace structures.

It turns out that Möbius and Laplace structures, although implicitly present in higher dimensions as well, are particularly important in dimensions 2 and 1 respectively, because they provide to an otherwise *flexible* conformal structure is such a low dimension the rigidity that is implicit in higher dimensions.

Remark 3.11. Obviously, any Möbius, resp. Laplace structure determine, through their principal symbols, the underlying conformal structure. The two propositions above show that, for higher dimensions, the converse also holds (but in a non-trivial way). In the next section we will see that any submanifold in a conformal m -dimensional manifold (M, c) inherits an *induced* Möbius structure (for $m \geq 3$) and an *induced* Laplace structure (if $m \geq 3$ or if $m = 2$ and M has, additionally, a Möbius structure).

Remark 3.12. The trace-free, resp. trace of (21) shows that, unless the weight k is equal to 1, resp. to $\frac{2-m}{2}$, the trace-free Hessian, resp. the trace of the Hessian can not be corrected merely by a scalar term in order to produce a conformally invariant operator (in general, a first-order correction term is also needed). This implies that the Möbius and the Laplace operators, as defined in Propositions 3.1 and 3.6 (including the specified conformal weights) are the only second-order, linear differential operators on weight bundles, that can be canonically associated to a conformal structure.

3.1. Geometric meaning. Möbius and Laplace structures have the following geometric interpretation: a non-vanishing section l of the corresponding weight bundle satisfies $\mathcal{P}l = 0$ if and only if the correction term in the defining formula from Proposition 3.1, resp. 3.6, when using the Levi-Civita connection ∇^l of the Riemannian metric g^l associated to l , vanishes identically, i.e.

$$\mathcal{M} = \text{Hess}_0^{\nabla^l}, \text{ resp. } \mathcal{L} = \text{tr}(\text{Hess}^{\nabla^l}).$$

This in turn means that $\mathcal{M}l = 0$ (for $n > 2$) if and only if the metric g^l is Einstein, and $\mathcal{L}l = 0$ (for $n > 1$) if and only if g^l is scalar-flat, which is a solution to the $s = 0$ *Yamabe problem*: finding metrics with (constant) scalar curvature equal to s (here $s = 0$) in the conformal class. The general Yamabe problem (finding metrics with constant, non-zero, scalar curvature in the conformal class) corresponds to the eigenvectors of \mathcal{L} (in order to identify L^k with L^{k-2} , so that the concept of eigenvalue makes sense, the solution l itself is used, which turns the general Yamabe problem into a non-linear one).

A little weaker than to find solutions to $\mathcal{M}l = 0$ is to look for Weyl structures, such that their trace-free Hessian is equal to \mathcal{M} (therefore, the trace-free symmetric Schouten-Weyl tensor (or, equivalently, Ricci tensor) vanishes); these connections are called *Einstein-Weyl structures*. For a Möbius surface, *Möbius-Einstein-Weyl structures* can be defined as Weyl structures ∇ such that $\mathcal{M} = \text{Hess}_0^\nabla$ (in [7], the author defines 2-dimensional Einstein-Weyl structures on a conformal surface as above, but requiring in addition that \mathcal{M} – which is not fixed in this case – is some *flat* Möbius structure, i.e., it is locally given by a conformal chart).

Analogously, the question of finding parametrizations (note that a metric on a curve is equivalent to a parametrization) of a given Laplace curve (C, \mathcal{L}) such that the zero-order term of \mathcal{L} , in this parametrization, is constant, can be seen as an extension to Laplace (or, equivalently, projective) curves of the Yamabe problem [4].

4. EXTRINSIC CONFORMAL GEOMETRY

In this section we define three conformally invariant tensors that characterize the embedding of two conformal manifolds, just like the second fundamental form does in the Riemannian setting. The goal is to define various geometric notions of *totally geodesic* conformal embeddings

(using the *conformal geodesics*, to be defined below) and characterize them by the vanishing of (some of) the above mentioned invariants.

4.1. Conformal geodesics. Let M^m , be a smooth manifold, endowed with a conformal structure if $m \geq 3$, a Möbius structure if $m = 2$ or a Laplace structure if $m = 1$. There is a way to define a *conformal acceleration* of an (immersed) parametrized curve in M :

Definition 4.1. Let $\gamma : I \rightarrow C \subset M$ be an immersion. Choose any Riemannian metric $g \in c$ and denote by h^g its Schouten tensor defined by (13) for $m > 2$, and by Proposition 3.4 for $m = 2$. Then the following vector field along the curve γ

$$a(\gamma) := g(\dot{\gamma}, \dot{\gamma})(h^g(\dot{\gamma}))^\sharp - \ddot{\gamma} + 3 \frac{g(\dot{\gamma}, \ddot{\gamma})}{g(\dot{\gamma}, \dot{\gamma})} \ddot{\gamma} + \left(-6 \frac{g(\dot{\gamma}, \ddot{\gamma})^2}{g(\dot{\gamma}, \dot{\gamma})^2} + \frac{3}{2} \frac{g(\ddot{\gamma}, \ddot{\gamma})}{g(\dot{\gamma}, \dot{\gamma})} + 2 \frac{g(\dot{\gamma}, \ddot{\gamma})}{g(\dot{\gamma}, \dot{\gamma})} \right) \dot{\gamma} \quad (22)$$

is called the *conformal acceleration* of γ . The notation $(h^g(\dot{\gamma}))^\sharp$ stands here for the vector field associated by the metric g to the 1-form $h^g(\dot{\gamma})$ along γ and $\ddot{\gamma} := \nabla_{\dot{\gamma}}^g \dot{\gamma}$, $\ddot{\gamma} := \nabla_{\dot{\gamma}}^g \ddot{\gamma}$.

It is well known (cf. [12, p. 67], [1, p. 218] or Lemma 4.8 below) that the conformal acceleration of a parametrized curve does not depend on the metric g .

The following observation is folklore (cf. e.g. [1]):

Lemma 4.2. *The normal part of $a(\gamma)$ does not depend on the parametrization γ of C .*

Conformal geodesics on a conformal manifold (resp. a Möbius surface or a Laplace curve) are then defined, by analogy with the notion of a geodesic of a connection, by the condition that the conformal acceleration vanishes [10], [11], [12]:

Definition 4.3. A regular curve $\gamma : I \rightarrow M$ in a conformal manifold (M^m, c) , with $m \geq 3$, a Möbius surface for $m = 2$, or a Laplace curve for $m = 1$, is called *parametrized conformal geodesic* if its conformal acceleration $a(\gamma)$ vanishes identically and *unparametrized conformal geodesic* if the normal part of the conformal acceleration vanishes.

The Cauchy-Lipschitz Theorem easily shows that every unparametrized conformal geodesic has local parametrizations turning it into a parametrized conformal geodesic.

Remark 4.4. The notion of conformal geodesic is also known in the literature to emerge out of the canonical *Cartan connection* associated to a given conformal n -manifold ($n \geq 3$), resp. a Möbius surface: The Cartan connection is a total parallelism

$$\omega : \mathbb{T}P \rightarrow \mathfrak{g},$$

on a certain principal bundle P over M , with values in the Lie algebra of the Möbius group $G := \mathrm{SO}(n+1, 1)$. The integral curves of the fundamental vector fields X^A on P , defined by elements $A \in \mathfrak{g}$, are parametrized curves on P that project on parametrized conformal geodesics on M in the sense of the Definition 4.3, see [8], [9].

Remark 4.5. E. Musso [17] uses the terminology “*conformal geodesics*” to denote some curves that are extremal with respect to a functional which consists of integrating the *conformal arc length* [18], [19] (which, in our terms, is the square root of the norm of the normal component of $a(\gamma)$) along the curve γ , assuming that this quantity is nowhere vanishing. Note that for our definition of the conformal geodesics, this “conformal arc length” vanishes identically, therefore Musso’s notion of “conformal geodesics” never coincides with the one we use here.

In fact, we rather consider Musso's "conformal arc length" to be a conformal analogon of the (*extrinsic*) *curvature* of the curve; the *arc length parametrizations* of a curve in a Riemannian manifold is rather an *intrinsic* structure on the curve, as is the induced *Laplace* (or projective) structure on a curve in a conformal manifold / Möbius surface (see below and, for more details, [4]).

Equivalently, the conformal geodesics can be defined as follows (see also [10], [12]): first, Corollary 4.13 below shows that every parametrized curve is geodesic for a certain Weyl structure which is then called *adapted*; then, the conformal, or Möbius, resp. Laplace geodesics can be defined as follows:

Definition 4.6. Let (M, c) be a conformal manifold of dimension at least 3, a Möbius surface or a Laplace curve. A regular curve $\gamma : I \rightarrow M$ is a parametrized *conformal*, resp. *Möbius geodesic* if any adapted Weyl structure ∇ (i.e., for which $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$) satisfies

$$h^{\nabla}(\dot{\gamma}) = 0 \quad (23)$$

on I . It is an unparametrized conformal geodesic if the section of ν induced by $h^{\nabla}(\dot{\gamma})$ vanishes identically.

Remark 4.7. The transformation rule (16) shows that the quantity $h^{\nabla}(\dot{\gamma})$ depends only on the restriction of the Weyl structure ∇ along the curve $\gamma(I)$. Note that no other similar tensor defined in the previous section ($h^{\nabla, s}$, σ^{∇} , Ric^{∇} , etc.) has this property.

The equivalence of this definition with the previous one is a consequence of

Lemma 4.8. *The conformal acceleration of a parametrized curve γ is equal to the vector field $c(\dot{\gamma}, \dot{\gamma})h^{\nabla}(\dot{\gamma})$ along γ , for any Weyl structure ∇ on (M, c) , adapted to γ .*

Proof. Choose any metric $g \in c$, and write $c = l^2 g$ for some section l of L . We denote by ∇^g the Levi-Civita connection of g and by $f^2 := g(\dot{\gamma}, \dot{\gamma})$. Then

$$c(\dot{\gamma}, \dot{\gamma}) = f^2 l^2, \quad (24)$$

and after two covariant derivatives with respect to $\dot{\gamma}$:

$$g(\ddot{\gamma}, \dot{\gamma}) = f f' \quad \text{and} \quad g(\ddot{\gamma}, \ddot{\gamma}) + g(\dot{\gamma}, \ddot{\gamma}) = f'^2 + f f'', \quad (25)$$

where we denoted by $f' := \dot{\gamma}(f)$ and $f'' := \dot{\gamma}(f')$.

Let $\nabla := \nabla^g + \tilde{\theta}$ be a Weyl structure on (M, c) and let $T := l^2 \theta$ be the dual vector field to θ by g . By (2) and (24), ∇ is adapted to γ if and only if

$$0 = \nabla_{\dot{\gamma}}^g \dot{\gamma} + 2\theta(\dot{\gamma})\dot{\gamma} - c(\dot{\gamma}, \dot{\gamma})\theta = \nabla_{\dot{\gamma}}^g \dot{\gamma} + 2l^{-2}g(T, \dot{\gamma})\dot{\gamma} - f^2 T.$$

Taking the scalar product with $\dot{\gamma}$ and using (25) we obtain $g(T, \dot{\gamma}) = -\frac{f'}{f}$. Reinjecting this in the same equation yields

$$T = \frac{\ddot{\gamma}}{f^2} - 2\frac{f'}{f^3}\dot{\gamma}. \quad (26)$$

Incidentally, this just proves the uniqueness of ∇ along γ , fact that we will prove in Proposition 4.12 to hold in a more general setting. From (25) and (26) we obtain

$$g(T, T) = \frac{g(\ddot{\gamma}, \ddot{\gamma})}{f^4}. \quad (27)$$

On the other hand, Equation (16) (with ∇' and ∇ there replaced by ∇ and ∇^g respectively) gives

$$h^\nabla = h^g - \nabla^g \theta + \theta \otimes \theta - \frac{1}{2} c(\theta, \theta) c.$$

Plugging $\dot{\gamma}$ in this formula and using (24) and (26) yields:

$$\begin{aligned} c(\dot{\gamma}, \dot{\gamma}) h^\nabla(\dot{\gamma}) &= c(\dot{\gamma}, \dot{\gamma}) \left(h^g(\dot{\gamma}) - \nabla_{\dot{\gamma}}^g \theta - \frac{f'}{f} \theta - \frac{1}{2} l^{-2} g(T, T) \dot{\gamma} \right) \\ &= f^2 l^2 \left(h^g(\dot{\gamma}) - l^{-2} \nabla_{\dot{\gamma}}^g \left(\frac{\ddot{\gamma}}{f^2} - 2 \frac{f'}{f^3} \dot{\gamma} \right) - \frac{f'}{f} \theta - \frac{1}{2} l^{-2} g(T, T) \dot{\gamma} \right) \\ &= f^2 \left((h^g(\dot{\gamma}))^\sharp - \nabla_{\dot{\gamma}}^g \left(\frac{\ddot{\gamma}}{f^2} - 2 \frac{f'}{f^3} \dot{\gamma} \right) - \frac{f'}{f} T - \frac{1}{2} \frac{g(\ddot{\gamma}, \ddot{\gamma})}{f^4} \dot{\gamma} \right). \end{aligned}$$

Developing this expression using (26) and the expressions for f' and f'' given by (25), proves the desired result. \square

4.2. Tensorial invariants of a conformal embedding. In the sequel, N , of dimension n , will be an embedded submanifold of a conformal manifold (M, c) , of dimension $m > n$, and we will consider N as a conformal submanifold of M endowed with the induced conformal structure still denoted by c . The cases when n or m are less than 3 will be specified explicitly.

We denote by ν the normal bundle to N , therefore

$$TM|_N = TN \oplus^\perp \nu.$$

We will denote by greek letters ζ, ξ , etc. normal vectors to N , and by capital roman letters X, Y , etc. tangent vectors to N . For a vector $A \in T_x M$, $x \in N$, the components of A in $T_x N$ and in ν_x will be denoted by A^N , resp. A^\perp .

For a given Weyl structure ∇ on M , the tangential component of $\nabla_X Y$, for $X, Y \in TN$ defines a Weyl structure on (N, c) , called the *induced* Weyl structure and denoted by ∇^N :

$$\nabla_X^N Y := (\nabla_X Y)^N.$$

The *normal* component of $\nabla_X Y$ (for $X, Y \in TN$) is the *fundamental form* of $N \subset M$ associated to ∇ , denoted by B^∇ :

$$B^\nabla : \otimes^2 TN \rightarrow \nu, \quad B^\nabla(X, Y) := (\nabla_X Y)^\perp.$$

For another Weyl structure $\nabla' = \nabla + \tilde{\theta}$, for θ a 1-form, we easily get

$$B^{\nabla'} - B^\nabla = -\theta^\perp \cdot c, \tag{28}$$

therefore we have the following well-known fact:

Proposition 4.9. *The trace-free part $B_0 \in S_0^2(T^*N) \otimes \nu$ of the fundamental form of a submanifold of a conformal manifold M depends only on the conformal structure c and the embedding $N \subset M$.*

A point of N where B_0 vanishes is called *umbilic*, and if $B_0 \equiv 0$, then N is a *totally umbilical* submanifold. For example, every curve in a conformal manifold is trivially totally umbilical. In general, (28) immediately implies

Proposition 4.10. *A conformal submanifold $N \subset M$ is totally umbilical iff there exists a Weyl structure ∇ on M , defined along N , such that N is ∇ -totally geodesic.*

The trace-part of the fundamental form is the *mean curvature*

$$H^\nabla \in \nu^* \simeq L^{-2} \otimes \nu, \quad H^\nabla(\xi) := -\frac{1}{n} \operatorname{tr}((\nabla \xi)^N),$$

such that we have the following decomposition of B^∇ :

$$B^\nabla = H^\nabla \cdot c + B_0. \quad (29)$$

By (28), the mean curvature transforms according to the rule

$$H^{\nabla'} - H^\nabla = -\theta^\perp. \quad (30)$$

A Weyl connection, resp. metric on (M, c) for which the mean curvature of N vanishes is called *adapted* to the embedding $N \subset M$. This property makes sense pointwise: recall that, in general, a conformal connection on a conformal manifold (M, c) is a section in an affine bundle (the associated vector bundle is $\Lambda^1 M \otimes \mathfrak{co}(M)$). A Weyl structure is a section in an affine subbundle, called $\operatorname{Weyl}(M)$ (the associated vector subbundle is the one of the 1-forms with values in $\mathfrak{co}(M)$ of type $\tilde{\theta}$, thus it is isomorphic with $\Lambda^1 M$).

Proposition 4.11. *Let $N \subset (M, c)$ be a submanifold in a conformal manifold. Then the adapting map*

$$\phi_N^M : \operatorname{Weyl}(N) \rightarrow \operatorname{Weyl}(M)|_N,$$

that extends each (pointwise) Weyl structure on N at x to an adapted Weyl structure on M at x is a right inverse to the restriction map

$$\operatorname{rest}_N^M : \operatorname{Weyl}(M)_N \rightarrow \operatorname{Weyl}(N).$$

Therefore, the image of the adapting map, $\operatorname{Weyl}(M)^N$, is isomorphic to $\operatorname{Weyl}(N)$ and the quotient $\operatorname{Weyl}(M)|_N / \operatorname{Weyl}^N$ is canonically isomorphic to the normal bundle ν .

The proof is a direct consequence of (28). For a result on adapted ambient metrics, a local viepoint is necessary:

Proposition 4.12. *Let $N \subset (M, c)$ be an embedded submanifold in a conformal manifold of dimension $m \geq 2$, and let H be a section in the dual normal bundle ν^* . For each metric g^N on N , there exists a metric g^M on M , inducing g^N , such that its Levi-Civita connection has H as the mean curvature of N . Any other such metric $g^{M'}$ satisfies*

$$g^{M'} = e^{2f} g^M, \quad \text{such that } f|_N \equiv 1 \text{ and } df|_N \equiv 0. \quad (31)$$

Proof. I It is easy to choose g^M such that it restricts to N as g^N , and denote by θ the difference between H and the mean curvature of N with respect to g^M . We need to show that there exists a function $f : M \rightarrow \mathbb{R}$ such that

- (1) $N = \{x \in M \mid f(x) = 0\}$
- (2) $df|_N = \theta$.

On a chart domain $U \simeq \mathbb{R}^m$, where $U \cap N \simeq \mathbb{R}^n \subset \mathbb{R}^m$ as the zero set of the coordinates y_1, \dots, y_{m-n} : then a section θ of $\nu^*|_{U \cap N}$ is given by

$$\theta(x) = \sum_{i=1}^{m-n} a_i(x) dy_i, \quad \forall x \in \mathbb{R}^n \simeq U \cap N$$

so the function $f^U : U \rightarrow \mathbb{R}$ defined by

$$f^U(x, y) := \sum_{i=1}^{m-n} a_i(x) y_i$$

satisfies $f^U|_{U \cap N} \equiv 0$ and $df^U|_{U \cap N} = \theta$.

Glueing such local functions using a covering of N by contractible charts and an underlying partition of unity yields a function $f : M \rightarrow \mathbb{R}$ such that $f|_N \equiv 0$ and $df|_N = \theta$, as required, thus the metric $g^M := e^{-2f}g$ restricts to g^N on N and its Levi-Civita connection $\nabla^M = \nabla^g + \tilde{df}$ induces the required mean curvature. \square

In the case of a curve in (M, c) , Propositions 4.11 and 4.12 show that

Corollary 4.13. *Every embedded curve C in a conformal-Riemannian manifold (M, c) is ∇ -geodesic for some suitable (adapted) Weyl structure on M . The restriction to $TM|_C$ of this Weyl structure is determined by a connection ∇^C on C (induced by ∇). If ∇^C is defined by a parametrization of C , ∇ can be chosen to be the Levi-Civita connection of a metric on M .*

In other words, every parametrized curve is a geodesic for a suitable metric.

Remark 4.14. In the pseudo-Riemannian conformal setting, this corollary still holds for embedded curves that are nowhere light-like.

We remark the following, concerning the connection induced on the normal bundle of N :

Proposition 4.15. *Let ∇ be a Weyl structure on M and ∇^N the induced Weyl structure on the submanifold N . Denote by ∇^\perp the connection induced by ∇ on the normal bundle ν of N in M :*

$$\nabla_X^\perp \xi := (\nabla_X \xi)^\perp, \quad \forall X \in TN. \quad (32)$$

Then ∇^\perp depends only on ∇^N . Moreover, the connection $\nabla^{\perp,0}$ induced on the weightless normal bundle $\nu^0 := \nu \otimes L^{-1}$ depends only on the embedding of N in the conformal manifold (M, c) . In particular, its curvature

$$\kappa \in \Lambda^2 \otimes \text{End}^{\text{skew}}(\nu) \quad (33)$$

is a tensor that depends on the conformal embedding alone.

Proof. Let $\nabla' = \nabla + \tilde{\theta}$, with θ a 1-form on M . Then $\nabla'^\perp = \nabla^\perp + \theta \otimes \text{Id}$, because the endomorphism of TN^\perp induced by $\theta \wedge X$ is trivial, for all $X \in TN$. The connection $\nabla'^{\perp,0}$ induced on $L^{-1} \otimes \nu$ coincides thus with $\nabla^{\perp,0}$. \square

We have seen before that the normal component of the ambient Schouten-Weyl tensor on an embedded curve (which is always totally umbilical), for an adapted Weyl structure, is an invariant that vanishes if and only if the curve is a conformal (resp. Möbius) geodesic: this invariant is the normal component of the conformal acceleration, see Lemma 4.8. We generalize this fact and define the *mixed Schouten-Weyl tensor of an embedding* as follows:

Proposition 4.16. *Let $N \subset M$ such that $m = \dim M \geq 3$ or $m = 2$ and M is endowed with a fixed Möbius structure \mathcal{M} . The mixed Schouten-Weyl tensor*

$$\mu : TN \rightarrow \nu^*, \quad \mu_X(\xi) := h^M(X, \xi) - (\nabla_X H^\nabla)(\xi) + \frac{1}{n-1}(\delta^\nabla B_0)(X)(\xi) \quad (34)$$

is independent of the Weyl structure ∇ on M used to compute the Schouten-Weyl tensor $h^M := h^\nabla$, the mean curvature H^∇ and the codifferential

$$(\delta^\nabla B_0)(X) := \text{tr}_{TN}^c ((\nabla \cdot B_0)(X, \cdot)),$$

Here ∇H^∇ and ∇B_0 are the covariant derivatives induced by ∇ on ν , resp. on $S_0^2 N \otimes \nu$. The convention for $n = 1$ is to omit the term in $\delta^\nabla B_0$ (B_0 is trivial anyway) in (34).

Proof. Proposition 4.15 implies that, for two Weyl structures $\nabla, \nabla' = \nabla + \tilde{\theta}$ that extend to M the same Weyl structure ∇^N on N (thus θ vanishes on TN), the connection used to compute the covariant derivative of the mean curvature does not change (but the mean curvature changes according to (30)). Moreover, the codifferential of B_0 (which is a conformal invariant itself) does not change at all.

By taking the difference of the tensors μ computed using ∇ , resp. ∇' in (34), the terms in the codifferential cancel, and from the terms in H , only a term in $\nabla(H^{\nabla'} - H^\nabla)$ remains, and it is equal to $-\nabla\theta$, according to (30).

On the other hand, using (16) for the vectors $X \in TN, \xi \in \nu$, we get

$$(h^{\nabla'} - h^\nabla)(X, \xi) = -\nabla_X \theta(\xi),$$

and this cancels with the term in ∇H as seen above.

Therefore, in computing μ via (34), we can choose Weyl structures adapted to the embedding. The expression then simplifies as the terms in H disappear.

Let ∇ and ∇' be adapted Weyl structures on M that satisfy (2), such that $\theta|_N$ is purely tangential. Using (16) and (30), we compute

$$\begin{aligned} (\delta^{\nabla'} B_0)(X) - (\delta^\nabla B_0)(X) &= \text{tr}_{TN}^c ((\tilde{\theta} \cdot B_0)(X, \cdot)) - \text{tr}_{TN}^c B_0(\tilde{\theta} \cdot X, \cdot) - \text{tr}_{TN}^c B_0(X, \tilde{\theta} \cdot) \\ &= (n-1)B_0(X, \theta) = (n-1)B^\nabla(X, \theta) \\ &= (n-1)(\nabla_X \theta)|_{\nu^*}, \end{aligned} \tag{35}$$

because the first term in the right hand side of the first line is equal to $B_0(X, \theta)$, the second vanishes and the third is equal to $(n-2)B_0(X, \theta)$. Note that the result is indeed a section of $\nu \otimes L^{-2} \simeq \nu^*$.

The claim follows now from (35) and (16) (for $n = 1$, the equation (35) is not needed, the claim follows directly from the reduction to adapted connections and (16)). \square

Of course, in a conformally flat space (locally isomorphic to the Möbius sphere), a totally umbilical submanifold of dimension at least 2 is a piece of a sphere, so the embedding is conformally equivalent to $\mathbb{R}^n \subset \mathbb{R}^m$. This means that the mixed Schouten-Weyl tensor automatically vanishes for totally umbilical submanifolds in a conformally flat space. The case where the ambient space is curved is different, as the following example shows:

Example 4.17. Cf. [3], a *conformal product* is a conformal manifold (M, c) such that a Weyl structure ∇ exists, that preserves an orthogonal pair of foliations, thus locally $M \simeq M_1^{m_1} \times M_2^{m_2}$. A simple example of a conformal product is given by the conformal class of a Riemannian product (with ∇ equal to the Levi-Civita connection of the product metric). However, if the conformal product is *non-closed* (the Weyl structure ∇ does not (even locally) preserve any metric, i.e. its Faraday form F is non-vanishing, then from Lemma 6.1 [3] we know that the

Schouten-Weyl tensor of ∇ , computed for X_1, X_2 each tangent to one of the foliations, has the following expression:

$$h^\nabla(X_1, X_2) = \frac{1}{2} \left(\frac{m_1 - m_2}{m_1 + m_2 - 2} - 1 \right) F(X_1, X_2). \quad (36)$$

Therefore, the leaves of each foliation are totally umbilical (because they are totally geodesic w.r.t. ∇), but the mixed Schouten-Weyl tensor is non-zero if $m_2 > 1$.

Here we see that, for $m_2 = 1$, the mixed Schouten-Weyl tensor of the first foliation vanishes. This is a general fact:

Proposition 4.18. *Let (M, c) be a conformal manifold of dimension at least 3. Then the mixed Schouten-Weyl tensor of any hypersurface $N \subset M$ is identically zero.*

Proof. This is a consequence of the Gauss and Codazzi equations: using the notations from Proposition 4.16, we have

$$\begin{aligned} c(R_{X,Y}^{\nabla^M} Z, \xi) &= c(\nabla_X B^{\nabla^M}(Y, Z) - \nabla_Y B^{\nabla^M}(X, Z), \xi), \\ c(R_{X,Y}^{\nabla^M} \xi, \zeta) &= c(R_{X,Y}^{\nabla^\perp} \xi, \zeta) + c(B^{\nabla^M}(X)(\xi), B^{\nabla^M}(Y)(\zeta)) - c(B^{\nabla^M}(Y)(\xi), B^{\nabla^M}(X)(\zeta)). \end{aligned}$$

Supposing that ∇^M is adapted and metric (so we have no Faraday form), (so $B^{\nabla^M} = B_0$) and taking the trace on TN in the first equation in Y and Z , we obtain

$$\text{Ric}_0^{\nabla^M}(X, \xi) = -\delta^{\nabla^M} B_0(X)(\xi),$$

because $\text{Ric}^{\nabla^M}(X, \xi) = \text{tr}_{TM} R_{\cdot, X}^{\nabla^M} \xi = \text{tr}_{TN} R_{\cdot, X}^{\nabla^M} \xi$, if TN^\perp is generated by ξ . For the Schouten-Weyl tensor of ∇^M , this implies

$$h^{\nabla^M}(X, \xi) = -\frac{1}{m-2} \delta^{\nabla^M} B_0(X)(\xi).$$

Using $n = m - 1$ (as N is a hypersurface), then (34) implies $\mu \equiv 0$. \square

Remark 4.19. Note that for $n = 1$ and $m = 2$, the computations above do not apply; in fact, for a curve N in a Möbius surface (M, c, \mathcal{M}) , the mixed Schouten-Weyl tensor is simply the normal part of the conformal (or rather Möbius) acceleration (Lemma 4.8), and thus vanishes if and only if the curve is a Möbius geodesic.

Another invariant derived from the Schouten-Weyl tensor is the *relative Schouten-Weyl tensor of an embedding*:

Proposition 4.20. *Let (M^m, c) be a conformal manifold of dimension $m \geq 3$ or a Möbius surface. Let $N^n \subset M^m$ such that $n = \dim N \geq 3$, or N is a Möbius surface with the same underlying conformal structure as the one induced by M , or a Laplace curve, and let ∇ be any Weyl structure on M . We denote as before by h^M and $H^\nabla \in \nu^*$ the Schouten-Weyl tensor of ∇ on M and the mean curvature of $N \subset M$ with respect to ∇ . Let h^N be the Schouten-Weyl tensor of N corresponding to the Weyl structure on N induced by ∇ (see Remarks 3.5 and 3.10 for the low-dimensional cases). Then the relative Schouten-Weyl tensor $\rho \in S^2 N$, defined by*

$$\rho(X, Y) := (h^M - h^N)(X, Y) + \frac{1}{2} c(H^\nabla, H^\nabla) c(X, Y) + H^\nabla(B_0(X, Y)), \quad \forall X, Y \in TN, \quad (37)$$

is independent of the Weyl structure ∇ .

Proof. As in the proof of Proposition 4.16, we treat first the case of two Weyl structures ∇ , $\nabla' = \nabla + \tilde{\theta}$ on M that restrict to the same connection ∇^N on N , i.e., $\theta|_{TN} \equiv 0$.

In the expressions (37) corresponding to ∇ , resp. ∇' , the terms in h^N are the same, and the difference of the terms in h^M is, cf. (16):

$$h^{\nabla'} - h^{\nabla} = -\nabla\theta + \theta \otimes \theta - \frac{1}{2}c(\theta, \theta)c, \quad (38)$$

where the middle term of the right hand side, restricted to $TN \otimes TN$, vanishes. The term in $\nabla\theta$ is, in fact, an expression in the fundamental form alone:

$$-\nabla_X\theta(Y) = \theta(\nabla_X Y) = c(\theta, H^\nabla)c(X, Y) + \theta(B_0(X, Y)), \quad \forall X, Y \in TN,$$

because $\theta(X) = \theta(Y) = 0$. We conclude

$$(h^{\nabla'} - h^{\nabla})|_{TN \otimes TN} = c(\theta, H^\nabla)c + \theta(B_0) - \frac{1}{2}c(\theta, \theta)c.$$

It is clear from (30) that the right hand side of the equation above cancels the difference

$$\left(\frac{1}{2}c(H^{\nabla'}, H^{\nabla'})c + H^{\nabla'}(B_0) \right) - \left(\frac{1}{2}c(H^\nabla, H^\nabla)c + H^\nabla(B_0) \right).$$

This means that, in the expression (37), we can restrict to adapted Weyl structures.

Let ∇ and $\nabla' = \nabla + \tilde{\theta}$ be adapted Weyl structures. Then the expressions (37) for ∇ , resp. ∇' , simplify and their difference is given by

$$(h^{\nabla'^M} - h^{\nabla^M})|_{TN \otimes TN} - (h^{\nabla'^N} - h^{\nabla^N}).$$

Applying (16) for both the pairs (∇'^M, ∇^M) and (the induced connections) (∇'^N, ∇^N) , we obtain the same result, because θ is tangential to N .

This shows the independence of ρ of the Weyl structure ∇ , as claimed. \square

Like μ , ρ vanishes on a totally umbilical submanifold of a conformally flat space. However, in the curved setting, the vanishing of B_0 , and even of μ , does not imply the vanishing of ρ :

Example 4.21. Let M be the Riemannian product $M_1 \times M_2$, and let $N := M_1 \times \{x\}$, where $x \in M_2$ (we assume $\dim M_i > 2$, $i = 1, 2$). N is totally geodesic in M , for the product metric (which is therefore adapted to the embedding), hence $B_0 = 0$, and, also for the product metric, μ vanishes. On the other hand, even if the Ricci tensor of N is the restriction to N of the Ricci tensor of M (again for the product metric), this property does not hold in general for the Schouten-Weyl tensors, because the normalizations depend on the scalar curvatures of M , resp. N , and on the dimensions m and n . For example, if M_1 is an Euclidean space and M_2 is a round sphere, $h^{M_1 \times M_2}|_N \neq h^{M_1} = 0$.

One can ask whether the Gauss and Codazzi type equations imply any further relation between the invariants B_0 , ∇^ν , μ and ρ of a conformal embedding $N \subset M$ (besides the vanishing of μ for hypersurfaces, cf. Proposition 4.18). The following Theorem proves the contrary: on any conformal manifold (or Möbius surface, or Laplace curve) one can prescribe all the invariant tensors defined in this section: B_0 , κ , μ^1 and ρ and realize them by a suitable conformal embedding:

¹for an embedding of N as a hypersurface, μ has to be zero, cf. Proposition 4.18

Theorem 4.22. *Let (N, g) be a Riemannian manifold and let ν be a vector bundle over N endowed with a metric g^ν and a connection ∇^ν . If $\dim N = 1$ or 2 , let N have a fixed Laplace, resp. Möbius structure, given by the Schouten-Weyl tensor h (associated to the Levi-Civita connection ∇ of g).*

Let $B_0 \in \mathcal{C}^\infty(S_0^2 N \otimes \nu)$ and $\rho \in \mathcal{C}^\infty(S^2 N)$ be given tensors; if $\text{rank } \nu > 1$ or if $\dim N = 1$ and $\text{rank } \nu = 1$, let $\mu \in \mathcal{C}^\infty(T^ N \otimes \nu^*)$ be a given arbitrary section, otherwise let μ be the zero section of this bundle.*

Then there exists a metric \tilde{g} on the total space M of ν , and, for $\dim N = \text{rank } \nu = 1$, there exists a Möbius structure on M , such that the fundamental form of N with respect to \tilde{g} is B_0 , and the mixed, resp. relative Schouten-Weyl tensors of N in M are equal to μ , resp. ρ .

Proof. It is enough to construct a metric \tilde{g} on a neighborhood of the zero section of the total space M of the vector bundle ν . We introduce the following tensors

$$a \in \mathcal{C}^\infty(T^* M \otimes \nu^*), \quad b \in \mathcal{C}^\infty(S^2 M), \quad f \in \mathcal{C}^\infty(M)$$

and we introduce the following notation: for each vector field X on M , we denote by \tilde{X} the horizontal (for the connection ∇^ν) lift on M of X . On the other hand, for each local section ζ in ν , we denote by $\tilde{\zeta}$ the vertical (local) vector field on M defined by ζ .

We have then, for each point $\xi \in M$, the following relations:

$$[\tilde{X}, \tilde{Y}]_\xi = \widetilde{[X, Y]}_\xi - \widetilde{R_{X,Y}^\nu \xi}, \quad [\tilde{X}, \tilde{\zeta}]_\xi = \widetilde{\nabla_X^\nu \zeta}, \quad [\tilde{\zeta}, \tilde{\eta}]_\xi = 0, \quad (39)$$

for each vector fields X, Y on M and sections ζ, η in ν .

We define the following symmetric bilinear form \tilde{g} on M : in a point $\xi \in M$ we define

$$\begin{aligned} \tilde{g}(\tilde{X}, \tilde{Y})_\xi &:= g(X, Y) - 2g^\nu(B_0(X, Y), \xi) + b(X, Y)\|\xi\|^2, \\ \tilde{g}(\tilde{X}, \tilde{\zeta})_\xi &:= \|\xi\|^2 a(X, \zeta), \\ \tilde{g}(\tilde{\zeta}, \tilde{\eta})_\xi &:= g^\nu(\zeta, \eta)(1 + f\|\xi\|^2). \end{aligned} \quad (40)$$

Here, $\|\xi\|^2 := g^\nu(\xi, \xi)$. It is clear that \tilde{g}_ξ is positive definite for ξ in a neighborhood of the zero section of ν .

We can compute the curvature of the Levi-Civita connection $\tilde{\nabla}$ of \tilde{g} on the points of $N \subset M$. For this, we first compute the covariant derivatives of vector fields of type $\tilde{X}, \tilde{\zeta}$ (as defined above) up to order 1 in the vertical (i.e., in the direction of the fibers of ν) directions:

$$\begin{aligned} \tilde{\nabla}_{\tilde{X}} \tilde{Y} &= \widetilde{\nabla_X Y} + B_0(X, Y) - b(X, Y)\xi - \frac{1}{2}R_{X,Y}^\nu \xi \\ &\quad - \nabla_X B_0(Y, \cdot)(\xi) - \nabla_Y B_0(X, \cdot)(\xi) + \nabla \cdot B_0(X, Y)(\xi) + \mathcal{O}(\|\xi\|^2), \\ \tilde{\nabla}_{\tilde{\zeta}} \tilde{X} &= -B_0(X, \cdot)(\zeta) + b(X, \cdot)g^\nu(\zeta, \xi) + \frac{1}{2}h(R_X^\nu \cdot \xi, \zeta) \\ &\quad + a(X, \cdot)g^\nu(\zeta, \xi) - a(X, \zeta)\xi + \mathcal{O}(\|\xi\|^2), \\ \tilde{\nabla}_{\tilde{X}} \tilde{\zeta} &= -B_0(X, \cdot)(\zeta) + b(X, \cdot)g^\nu(\zeta, \xi) + \frac{1}{2}g^\nu(R_X^\nu \cdot \xi, \zeta) \\ &\quad + a(X, \cdot)g^\nu(\zeta, \xi) - a(X, \zeta)\xi + \nabla_X^\nu \zeta + \mathcal{O}(\|\xi\|^2), \\ \tilde{\nabla}_{\tilde{\zeta}} \tilde{\eta} &= (g^\nu(\xi, \zeta)\eta + g^\nu(\xi, \eta)\zeta - g^\nu(\zeta, \eta)\xi)f + a(\cdot, \eta)g^\nu(\xi, \zeta) + a(\cdot, \zeta)g^\nu(\xi, \eta) + \mathcal{O}(\|\xi\|^2). \end{aligned} \quad (41)$$

We compute then the curvature of $\tilde{\nabla}$ on N and the Ricci tensor and the scalar curvature is then, for $x \in N \subset M$:

$$\begin{aligned}\widetilde{\text{Ric}}(\tilde{X})_x &= \widetilde{\text{Ric}}(X) - (m-n)b(X, \cdot) - \delta^\nabla B_0(X) - (m-n-1)a(X, \cdot) \\ \widetilde{\text{Ric}}(\tilde{\zeta})_x &= -\delta^\nabla B_0(\cdot)(\zeta) - (m-n-1)a(\cdot, \zeta) - \text{tr}(b)\zeta - 2(m-n-1)f\zeta. \\ \widetilde{\text{Scal}} &= \text{Scal} - 2(m-n)\text{tr}(b) - 2(m-n-1)(m-n)f.\end{aligned}\quad (42)$$

If $m-n-1 > 0$, the map associating the mixed Schouten-Weyl tensor μ of the embedding $N \subset M$ to a tensor a is an invertible affine map.

If $m \geq 2$, the map associating the trace of the relative Schouten-Weyl tensor ρ to f is also an invertible affine map.

Once a, f and the trace part of b are fixed, if $m \geq 3$, the map associating the trace-free relative Schouten-Weyl tensor ρ_0 to b_0 (the trace-free part of b) is also an invertible affine map.

In conclusion, for any given tensors μ and ρ as above, by choosing a, b and f , the mixed and relative Schouten-Weyl tensors of $N \subset (M, [\tilde{g}])$ are precisely μ , resp. ρ .

The same conclusion holds in the special case $m = 2$ (when M is a Möbius surface) and $n = 1$ (N is a Laplace curve), but in this case \tilde{g} can be arbitrarily chosen; the mixed Schouten-Weyl tensor μ and the relative Schouten-Weyl tensor ρ determine (and are determined by) a choice of a Möbius structure on M , or, equivalently, by a choice of a trace-free symmetric Schouten-Weyl tensor \tilde{h}_0 . Therefore, any given tensors μ and ρ can be realized as the corresponding invariants of an embedding of the Laplace curve N in some Möbius surface M . \square

4.3. Induced Möbius and Laplace structures on submanifolds.

4.3.1. Induced Möbius structure. Let $N^n \subset M^m$ be a submanifold in the conformal manifold (M, c) with $m > n \geq 2$. We want to construct an *induced* Möbius structure on N (see also [6]):

Take any Weyl structure ∇ on (M, c) . We try to mimic the definition of the canonical Möbius structure by considering the Hessian of ∇ acting on vectors of TN and on sections of L_N . Since the usual Hessian possibly involves a covariant derivative in normal directions, we need to modify it by suppressing this derivative. More precisely, we define the “horizontal” Hessian acting on sections of L_N by

$$\overline{\text{Hess}}^\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_{(\nabla_X Y)^N} = \text{Hess}^\nabla(X, Y) + \nabla_{B(X, Y)}, \quad \forall X, Y \in TN. \quad (43)$$

We define then the second order operator $P : C^\infty(L_N) \rightarrow C^\infty(T^*N \otimes T^*N \otimes L_N)$ by

$$P(X, Y)l = \overline{\text{Hess}}^\nabla(X, Y)l + h^\nabla(X, Y)l, \quad \forall X, Y \in TN, \quad l \in C^\infty(L_N).$$

We now study the way P changes under a change of Weyl structure. If we replace ∇ by $\nabla' = \nabla + \tilde{\theta}$, the formulas for the change of the Schouten-Weyl tensor (16) and of the Hessian (21), together with (28) yield

$$\begin{aligned}P'(X, Y)l - P(X, Y)l &= c(X, Y)(\nabla_\theta l + \frac{1}{2}c(\theta, \theta)l) + \nabla'_{B'(X, Y)}l - \nabla_{B(X, Y)}l \\ &= c(X, Y)(\nabla_{\theta^N} l + \frac{1}{2}c(\theta, \theta)l - c(\theta, \theta^\perp)l) + \theta(B(X, Y))l.\end{aligned}$$

Let P_0 denote the trace-free part of P : $P_0(X, Y)l = P(X, Y)l - \frac{1}{n}c(X, Y)\text{tr}_{TN}(Pl)$. The previous relation yields

$$P'_0(X, Y)l - P_0(X, Y)l = \theta^\perp(B_0(X, Y))l. \quad (44)$$

We conclude:

Proposition 4.23. *Let $N^n \subset (M^m, c)$ be conformal submanifold, with $m > n \geq 2$. The second order operator*

$$\mathcal{M}^{ind} : C^\infty(L_N) \rightarrow C^\infty(T^*N \otimes T^*N \otimes L_N)$$

defined by

$$\mathcal{M}^{ind}(X, Y)l = P_0(X, Y)l + H^\nabla(B_0(X, Y))l, \quad (45)$$

or, equivalently, by

$$\mathcal{M}_{X,Y}^{ind}l := (\text{Hess}_{0,N}^\nabla(X, Y)l + \nabla_{B_0(X,Y)}l) + h_{0,N}^\nabla(X, Y) + H^\nabla(B_0(X, Y))l \quad (46)$$

(here the double subscript $0, N$ means the trace-free part along TN) does not depend on the choice of the Weyl structure ∇ , and defines a Möbius structure on N , called the induced Möbius structure on N .

Moreover, if $n \geq 3$ or $n = 2$ and N has a Möbius structure \mathcal{M}^N compatible with c , we have

$$\mathcal{M}^N - \mathcal{M}^{ind} = \rho_0, \quad (47)$$

where ρ is the relative Schouten-Weyl tensor of the embedding (37), and ρ_0 its trace-free part.

Proof. The independence of \mathcal{M}^{ind} of the Weyl structure ∇ follows directly from (44) and (30).

We will now compute the difference between this induced Möbius structure and the Möbius structure of N . Since the horizontal Hessian $\overline{\text{Hess}}^\nabla(X, Y)l$ appearing in the definition of P is just the Hessian of the induced Weyl structure on N applied to l , we get

$$\begin{aligned} P(X, Y)l + H^\nabla(B(X, Y))l - (\text{Hess}^N(X, Y)l + h^N(X, Y)l) = \\ h^M(X, Y)l - h^N(X, Y)l + H^\nabla(B(X, Y))l. \end{aligned}$$

The trace-free part of the left hand side equals $\mathcal{M}^{ind}l - \mathcal{M}^Nl$, whereas the trace-free part of the right hand side is just the trace-free part of ρ . This proves our claim. \square

4.3.2. Induced Laplace structure. Let $N^n \subset (M^m, c)$ be conformal submanifold, with $m > n \geq 1$, and if $m = 2$, suppose M has a Möbius structure (see also [6]).

In order to define the induced Laplace structure on N , we take, by analogy with the definition in Proposition 3.6, the trace on TN of the horizontal Hessian acting on sections of L^k for $k = 1 - \frac{n}{2}$ and add k times the trace over TN of the Schouten tensor h^∇ of M with respect to ∇ . The resulting operator, say $D : C^\infty(L^k) \rightarrow C^\infty(L^{k-2})$, defined by

$$D(l) := \text{tr}_{TN}(\overline{\text{Hess}}^\nabla l + kh^\nabla(\cdot, \cdot))l$$

is not independent on ∇ . Indeed, for a new Weyl structure $\nabla' = \nabla + \tilde{\theta}$, we get using (16), (21) and (43):

$$\begin{aligned} D'(l) - D(l) &= \text{tr}_{TN}(\overline{\text{Hess}}^{\nabla'} l - \overline{\text{Hess}}^{\nabla} l + kh^{\nabla'}(\cdot, \cdot)l - kh^{\nabla}(\cdot, \cdot)l) \\ &= n\nabla'_{H^{\nabla'}} l - n\nabla_{H^{\nabla}} l + 2(k-1)\nabla_{\theta^N} l + n\nabla_{\theta} l \\ &\quad + k[-\delta^N \theta + (k-2)c(\theta^N, \theta^N) + nc(\theta, \theta)]l \\ &\quad + k[\delta^N \theta + c(\theta^N, \theta^N) - \frac{n}{2}c(\theta, \theta)]l. \end{aligned}$$

After simplifications (using (4), (30) and the choice $k = 1 - \frac{n}{2}$), we get

$$D'(l) - D(l) = kn c(\theta, H^{\nabla'})l + \frac{kn}{2}c(\theta^{\perp}, \theta^{\perp})l.$$

On the other hand, by (30) the expression $c(H^{\nabla}, H^{\nabla})$ transforms according to the rule

$$c(H^{\nabla'}, H^{\nabla'}) - c(H^{\nabla}, H^{\nabla}) = -2c(\theta^{\perp}, H^{\nabla'}) - c(\theta^{\perp}, \theta^{\perp}).$$

We have proved the following:

Proposition 4.24. *Let $N^n \subset (M^m, c)$ be conformal submanifold, with $m > n \geq 1$, and if $m = 2$, suppose M has a Möbius structure. The following operator is well-defined, and is independent of the Weyl structure ∇ :*

$$\begin{aligned} \mathcal{L}^{ind} : \mathcal{C}^{\infty}(L_N^{1-n/2}) &\rightarrow \mathcal{C}^{\infty}(L_N^{-1-n/2}), \\ \mathcal{L}^{ind} l &:= \text{tr}_{TN}(\overline{\text{Hess}}^{\nabla} l) + k \text{tr}_{TN} h^{\nabla} l + \frac{kn}{2}c(H^{\nabla}, H^{\nabla})l. \end{aligned} \quad (48)$$

It is called the induced Laplace structure on N .

There is a geometric interpretation of the induced Möbius operator: Suppose l is a section in L over the submanifold M , that does not vanish. The square of l is the a metric g^N on N in the conformal class c . It is actually a metric on M as well, but it is only defined *along* N . We can, however, extend it to a metric g^M on (a neighborhood of N in) M , and we call such an extension *minimal* if (N, g^N) is a minimal submanifold of (M, g^M) . We have then a corresponding *minimal* extension l^{min} of l as a section of L over M and it holds:

Proposition 4.25. *For a minimal extension of a non-vanishing section l of L over N to a section l^{min} of L over M , we have*

$$\mathcal{M}^{ind} l = \mathcal{M}^M l^{min}.$$

5. CONFORMAL GEODESIC SUBMANIFOLDS

In Riemannian geometry, an embedding $N \subset (M, g)$ is *totally geodesic* if and only if one of the following equivalent properties hold:

- (1) The fundamental form B of $N \subset M$ vanishes
- (2) For every M -geodesic $\gamma : I \rightarrow M$ ($I \subset \mathbb{R}$ contains a neighborhood of 0) that is defined by initial conditions

$$\gamma(0) = x, \quad \dot{\gamma}(0) = X, \quad (49)$$

with $x \in N$ and $X \in T_x N$, there exists $\varepsilon > 0$ such that $\gamma(-\varepsilon, \varepsilon)$ is contained in N

- (3) All N -geodesics are also M -geodesics.

In conformal geometry, these three properties give rise to different notions: The first is the one of a *totally umbilical* submanifold, that we repeat here:

Definition 5.1. A submanifold $N^n \subset (M^m, c)$ in a conformal manifold is *totally umbilical* if and only if the trace-free fundamental form B_0 of the embedding vanishes identically.

The second notion gives rise to the notion of a *weakly geodesic submanifold*:

Definition 5.2. A submanifold $N^n \subset (M^m, c)$ in a conformal manifold with $\dim M \geq 3$ or a Möbius surface is *weakly geodesic* if and only if, for every initial conditions $x \in N$, $X \in T_x N$ and $D \in \text{Weyl}_x^N$ for the conformal geodesic equation (23) in M :

$$\gamma : I \rightarrow M, \quad \gamma(0) = x, \quad \dot{\gamma}(0) = X, \quad \nabla_x = D, \quad (50)$$

where I is an interval that contains a neighborhood of 0, there exists $\varepsilon > 0$ such that $\gamma(-\varepsilon, \varepsilon)$ is contained in N .

Finally, the strongest notion of a conformal geodesic submanifold is

Definition 5.3. Let $N^n \subset (M^m, c)$ be a submanifold of dimension at least 3, or $\dim N = 2$ and N has a compatible Möbius structure, or $\dim N = 1$ and N has a compatible Laplace structure. Suppose $\dim M \geq 3$ or $\dim M = 2$ and M has a Möbius surface compatible with c . N is then *strongly geodesic* if and only if every N -conformal (or Möbius, or Laplace) geodesic is also an M -conformal or Möbius geodesic.

Theorem 5.4. For a submanifold $N \subset M$, where (M, c) is a conformal manifold of dimension $m \geq 3$ or a Möbius surface, and N has, if $\dim N = 1$ or 2, a Laplace, resp. Möbius structure compatible with c , the following implications hold:

$$\text{strongly geodesic} \Rightarrow \text{weakly geodesic} \Rightarrow \text{totally umbilical}, \quad (51)$$

and the converse implications hold if (M, c) is conformally flat (for $m \geq 3$), resp. (M, c, \mathcal{M}) is Möbius flat (for $m = 2$). Moreover, these three properties of the embedding $n \subset M$ are equivalent to the vanishing of the following invariant tensors:

- (1) $N \subset M$ is totally umbilical $\iff B_0 \equiv 0$
- (2) $N \subset M$ is weakly geodesic $\iff B_0 \equiv 0$ and $\mu \equiv 0$
- (3) $N \subset M$ is strongly geodesic $\iff B_0 \equiv 0$, $\mu \equiv 0$ and $\rho \equiv 0$.

Proof. Let $N \subset M$ be weakly geodesic. Fix $x \in N$ and fix an adapted Weyl structure ∇ at x . Consider now all conformal geodesics that satisfy the initial conditions (50) for all $X \in T_x N$. These curves all lie in N , thus in particular we have

$$\nabla_X X = 0, \quad \forall X \in T_x N.$$

This means, in particular, as x was chosen arbitrarily, that N is totally umbilical. Moreover, for each conformal geodesic satisfying the initial conditions (50), $h^\nabla(X, \xi) = 0$ for some local extension of the Weyl structure ∇ and for any normal vector $\xi \in \nu_x$. In particular $\mu_x = 0$, which proves the claim (2).

Suppose now that N is strongly geodesic. In particular, all the conformal geodesics γ^X satisfying the initial conditions (50), for any $x \in N$ fixed and a fixed connection D on N , are

also conformal geodesics in M . However, the initial conditions satisfied by γ^X as a conformal geodesic on M are

$$\gamma^X(0) = x, \dot{\gamma}^X(0) = X \text{ and } \nabla_{\gamma^X}^{\gamma^X} = \nabla^X \in \text{Weyl}^M(x), \quad (52)$$

where we don't know yet what is the Weyl structure ∇^X , except that it is adapted to the parametrized curve γ in M , thus it has to be an extension of the connection D on N , with respect to which γ is geodesic in N . We will simply relate it to the adapted extension ∇ of D as

$$\nabla^X = \nabla + \tilde{\theta}^X, \text{ where } \theta^X \in \nu^*.$$

The mean curvature of ∇^X at x is then $-\theta^X$ (30). On the other hand, $(\nabla_X^X X)_x = 0$, because γ^X is geodesic for ∇^X , therefore

$$-\theta^X c(X, X) + B_0(X, X) = 0. \quad (53)$$

Now we compute the relative Schouten-Weyl tensor ρ at x using the connection ∇^X :

$$\rho_x = h_x^{\nabla^X}|_{T_x N \times T_x N} - h_x^D + \frac{1}{2}c(\theta^X, \theta^X)c - \theta^X(B_0).$$

Because γ^X is a conformal geodesic at x for both D and ∇^x , the Schouten-Weyl tensors of D and ∇^X vanish in the direction of X : As this holds for all vectors $X \in T_x N$, this means that $h_x^D = 0$, and

$$\rho(X) = \frac{1}{2}c(\theta^X, \theta^X)X - \theta^X(B_0(X)), \quad \forall X \in T_x N, \quad (54)$$

which, cf. (53), implies

$$\rho(X, X)c(X, X) = -\frac{1}{2}c(B_0(X, X), B_0(X, X)), \quad \forall X \in T_x N. \quad (55)$$

We compute now the mixed Schouten tensor using the connection ∇^X (we can suppose $n > 1$ since the 1-dimensional case is trivial):

$$\mu(X) = h^{\nabla^X}(X)|_{\nu} + \nabla_X \theta^X + \frac{1}{n-1}(\delta^{\nabla} B_0)(X).$$

Of course, $h^{\nabla^X}(X)|_{\nu} = 0$, hence we obtain

$$\mu_x(X) = \nabla_X B_0(X, X)/c(X, X) + \frac{1}{n-1}(\delta^{\nabla} B_0)(X), \quad \forall X \in T_x N \setminus 0. \quad (56)$$

Recall that the connection used to compute ∇B_0 and $\delta^{\nabla} B_0$ depends only on the restriction to N of this connection (Proposition 4.15). Because μ and $\delta^{\nabla} B_0$ are linear maps from $T_x N$ to ν_x^* , the same must hold for the map

$$X \mapsto \frac{\nabla_X B_0(X, X)}{c(X, X)},$$

i.e., there exists a tensor $a \in T^*M \otimes \nu$ such that

$$\nabla_X B_0(X, X) = a(X)c(X, X), \quad \forall X \in TM. \quad (57)$$

Now, the tensor ∇B_0 is a 3-tensor with values in ν , that is symmetric and trace-free in the last 2 arguments. There is, therefore, only one trace left, $\delta^{\nabla} B_0$. The equation (57) shows that the symmetric part of this 3-tensor is equal to the symmetric part of $a \otimes c$, more precisely

$$\nabla_X B_0(Y, Z) + \nabla_Y B_0(X, Z) + \nabla_Z B_0(X, Y) = a(X)c(Y, Z) + a(Y)c(X, Z) + a(Z)c(Y, X),$$

for all $X, Y, Z \in TN$. By taking the trace in Z, Y , we obtain

$$2\delta^\nabla B_0(X) = (n+2)a(X),$$

Thus (56) becomes

$$\mu = \frac{3n}{n-1}\delta^\nabla B_0. \quad (58)$$

We have started with one Weyl structure $D \in \text{Weyl}_x(N)$, but the whole argument holds for any starting Weyl structure, thus the relation (58) must hold for $D^\eta := D + \tilde{\eta}$, for all $\eta \in T_x^*N$. Equation (35), together with (58), implies then

$$3nB_0(\eta, X) = 0, \quad \forall X \in T_xN, \forall \eta \in T_x^*N,$$

thus $B_0 \equiv 0$ and N is totally umbilical. In particular, (53) implies that all the connections ∇^X coincide with ∇ at x and thus, all the conformal geodesics of N are conformal geodesics for M corresponding to adapted initial conditions. In other words, $N \subset M$ is weakly geodesic.

From (58) and (55) we also conclude that, for a strongly geodesic submanifold $N \subset M$, B_0 , μ and ρ vanish identically.

Conversely, if $B_0 \equiv 0$ and $\mu \equiv 0$, we consider on N the following type of curves: an *adapted conformal geodesic* on N is a smooth curve $\gamma : I \rightarrow N$ such that, for the adapted Weyl structure D on it, $\bar{h}(\dot{\gamma}) = 0$, where $\bar{h} := h^\nabla|_{TN \otimes TN}$, where ∇ is the unique adapted extension to M of the Weyl structure D (by Proposition 4.15, ∇ is also defined along the curve $\gamma(I)$, like D). This third order ODE has solutions on N , and because $B_0 \equiv 0$ and $\mu \equiv 0$, the resulting curves are conformal geodesics in M . N is, thus, weakly geodesic.

If, additionally, $\rho \equiv 0$, the above construction yields precisely the conformal geodesics of N , thus N is strongly geodesic in M , as claimed. \square

One can ask whether the hypothesis of weak or strong geodesy can be relaxed, for example:

Definition 5.5. A submanifold $N \subset M$ for which, in each point, the conformal geodesics of N with the initial conditions (50) for all $x \in N$, all $X \in T_xN$, for some $D \in \text{Weyl}_x(N)$, are conformal geodesics in M , is called *pseudo-geodesic*.

It is easy to see that a pseudo-geodesic curve is automatically strongly geodesic.

From the proof of the above Theorem, it is clear that a pseudo-geodesic submanifold N is strongly geodesic in M , provided N is totally umbilical. The following example shows that, in higher dimensions, there are non-umbilic submanifolds that are pseudo-geodesic:

Example 5.6. Let $N := \mathbb{R}^2$ with the flat metric and the flat Möbius structure, and consider ν the trivial vector bundle of rank 2 on \mathbb{R}^2 . Let B_0 be defined as follows:

$$B_0(\partial_x, \partial_y) = V, \quad B_0(\partial_x, \partial_x) = -B_0(\partial_y, \partial_y) := W,$$

for V, W an orthonormal frame of ν . Let ∇^ν be a unitary connection on ν such that V and W are parallel. Define the tensor ρ as follows:

$$\rho(\partial_x, \partial_x) = \rho(\partial_y, \partial_y) := -\frac{1}{2}, \quad \rho(\partial_x, \partial_y) := 0.$$

Then, by Theorem 4.22, define a metric \tilde{g} on $M \subset \mathbb{R}^4$ by the conditions $\mu \equiv 0$ and ρ and B_0 as above.

For the Levi-Civita connection D on N given by the flat euclidean metric, the geodesics (and Möbius geodesics as well) are clearly the affine lines. Let ∇ be the adapted Weyl structure on M for $N \subset M$ (in fact, it is just the Levi-Civita connection of \tilde{g} , as given by Theorem 4.22).

Consider now, for each D -parallel unitary vector field $X^t := \cos t \partial_x + \sin t \partial_y$, we consider the Weyl structure

$$\nabla^t := \nabla + \tilde{\theta}^t, \text{ where } \theta^t := \cos(2t)W + \sin(2t)V.$$

For future use, we compute

$$\langle X^t, X^s \rangle = \cos(t - s), \quad B_0(X^t, X^s) = \theta^{\frac{t+s}{2}}, \quad \langle \theta^t, \theta^s \rangle = \cos(2t - 2s). \quad (59)$$

For the connection ∇^t , we have

$$\nabla_{X^t}^t X^t = B_0(X^t, X^t) - \cos(2t)W - \sin(2t)V = 0,$$

thus ∇^t is the connection adapted to the affine line generated by X^t . In order to compute $h^t(X^t)$, for h^t the Schouten-Weyl tensor of ∇^t on M , we use ∇^t , resp. D in the defining formulas for μ and ρ and obtain

$$h^t(X^t, V) = h^t(X^t, W) = 0,$$

and

$$h^t(X^t, X^s) = \rho(X^t, X^s) - \frac{1}{2} \langle X^t, X^s \rangle + \langle \theta^t, B_0(X^t, X^s) \rangle = 0.$$

Therefore, the affine lines in the direction X^t in N are conformal geodesics in M , with adapted Weyl structure ∇^t . N is thus spanned by a large family of common conformal geodesics (for N and M), however, N is nowhere umbilic in M .

It is possible to construct such examples in higher dimensions as well. On the other hand, it is clear from (54) that for $X, Y \in T_x N$, $X \perp Y$, we get

$$\theta^X(B_0(X, Y)) = \theta^Y(B_0(X, Y)),$$

thus a pseudo-geodesic Möbius surface in a 3-manifold is always totally umbilic, thus strongly geodesic, so the example above realizes the minimal dimensions (m, n) for which a non-umbilic, pseudo-geodesic submanifold of dimension n is a m -dimensional conformal manifold exists.

It would be interesting to find for which dimensions (m, n) the pseudo-geodesic n -submanifolds of a conformal m -manifold are automatically umbilical.

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